MATH 3310 Tutorial

2020 ~ 2021

Term 2

(1) Let $E(u) = \int_0^1 \left(\frac{du}{dx}\right)^4 dx$, where $u : [0,1] \to \mathbb{R}$ smooth, $u(0) = u(1) = 0$.

How can we find a u such that $E(u)$ is minimized?

We know how to minimize a real-valued function using calculus. But here, we cannot use calculus directly because the input u of $E(u)$ is a function, not a real number.

A smart trick:

Suppose u is a local minimizer of E.

In this case, for any variation function $v : [0,1] \to \mathbb{R}$ with $v(0) = v(1) = 0$, we know

$$E(u) \le E(u+tv) \quad \text{if } |t| \in \mathbb{R} \text{ is small enough}$$

Rmk: $(u+tv)(0) = (u+tv)(1) = 0$ since $v(0) = v(1) = 0$.

Now, we define $G(t) := E(u+tv) = \int_0^1 \left(\frac{du}{dx} + t\frac{dv}{dx}\right)^4 dx$, $t \in \mathbb{R}$.

Since $E(u)$ is a local minimum of E,

we know $G(0)$ is a local minimum of $G(t)$.

Here, $G(t)$ is a real-valued function, which means we can take derivative.

So we have $\left.\frac{d}{dt}\right|_{t=0} G(t) = G'(0) = 0$.

$$= \left.\frac{d}{dt}\right|_{t=0} \int_0^1 \left(\frac{du}{dx} + t\frac{dv}{dx}\right)^4 dx$$

$$= \int_0^1 \left.\frac{d}{dt}\right|_{t=0} \left(\frac{du}{dx} + t\frac{dv}{dx}\right)^4 dx$$

$$= \int_0^1 4\left(\frac{du}{dx} + t\frac{dv}{dx}\right)^3 \cdot \frac{dv}{dx}\Big|_{t=0} dx$$

$$= \int_0^1 4\left(\frac{du}{dx}\right)^3 \frac{dv}{dx}\, dx$$

$$= \int_0^1 4\left(\frac{du}{dx}\right)^3 dv$$

$$= \underbrace{4v\left(\frac{du}{dx}\right)^3\Big|_{x=0}^{x=1}}_{=0 \text{ since } v(0)=v(1)=0} - \int_0^1 4v\, d\left(\frac{du}{dx}\right)^3 \quad \text{(Integration by part)}$$

$$= -\int_0^1 12\left(\frac{du}{dx}\right)^2 \frac{d^2u}{dx^2}\, v\, dx$$

$= 0$ for any $v: [0,1] \to \mathbb{R}$ with $v(0) = v(1) = 0$.

$$\Rightarrow \left(\frac{du}{dx}\right)^2 \frac{d^2u}{dx^2} = 0 \text{ on } [0,1].$$

(2) Let Ω be a smooth compact domain in $\mathbb{R}^n$.

Define $E(u) = \int_\Omega |\nabla u|^2 dx$, where $u \in C^\infty(\Omega)$ and $u = 0$ on $\partial\Omega$.

How can we find local minimizers of E?

Let v be a smooth function in Ω with $\underline{v = 0}$ on $\partial\Omega$.

If u is a local minimizer,

then consider $G(t) = \int_\Omega |\nabla(u+tv)|^2 dx$.

So, $\frac{d}{dt}\Big|_{t=0} G(t) = \int_\Omega 2\nabla u \cdot \nabla v + \underbrace{t^2|\nabla v|^2}_{=0} dx$

$$= 2\int_\Omega \nabla u \cdot \nabla v\, dx$$

$$= \underbrace{2\int_{\partial\Omega} v(\nabla u \cdot \vec{n})\, dS_x}_{=0} - 2\int_\Omega v \Delta u\, dx$$

if $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$

$$\Delta u = \frac{\partial^2 u_1}{\partial x_1^2} + \cdots + \frac{\partial u_n^2}{\partial x_n^2}$$

$$= -2\int_\Omega v \Delta u \, dx = 0$$

$$\Rightarrow \Delta u = 0 \quad \text{in } \Omega$$

1. Find a solution to $y' + x^2 y = 2x^2 e^{\frac{2x^2}{3}}$, $y(0) = 3$

$$M(x) = e^{\int x^2 dx} = e^{\frac{1}{3}x^3}$$

Then, multiply $M(x)$ on both sides of the differential equation.

$$e^{\frac{1}{3}x^3}(y' + x^2 y) = 2x^2 e^{x^3}$$

$$\frac{d}{dx}\left(e^{\frac{1}{3}x^3} y\right) = 2x^2 e^{x^3}$$

$$\Rightarrow e^{\frac{1}{3}x^3} y = \int 2x^2 e^{x^3} dx + C$$

$$= \frac{2}{3} e^{x^3} + C$$

$$\Rightarrow y = \frac{1}{3} e^{\frac{2}{3}x^3} + \frac{C}{e^{\frac{1}{3}x^3}}$$

$$y(0) = \frac{1}{3} + C = 3 \Rightarrow C = \frac{8}{3}$$

2. Let $f(x) = x$ on $[-\pi, \pi]$, of period 2π.

Calculate the Fourier series of f.

We intend to write $f(x) = a_0 + \sum_{i=1}^{\infty} a_n \cos nx + b_n \sin nx$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x \, dx$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \frac{1}{n}\, d\sin nx = \frac{1}{\pi}\,\frac{x}{n}\sin nx\Big|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi}\sin nx\, dx$$

$$= -\frac{1}{n\pi}\left(-\frac{1}{n}\cos nx\right)\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{n^2\pi}(-1)^n - \frac{1}{n^2\pi}(-1)^{-n} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\, \frac{-1}{n}\, d\cos nx$$

$$= -\frac{x}{n\pi}\cos nx\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi}\cos nx\, dx$$

$$= -\frac{\pi}{n\pi}(-1)^n + \frac{-\pi}{n\pi}(-1)^{-n} + \frac{1}{n\pi}\,\frac{\sin nx}{n}\Big|_{-\pi}^{\pi}$$

$$= -\frac{2}{n}(-1)^n + 0 = -\frac{2}{n}(-1)^n$$

3. Calculate the Fourier series of x^2 on $[-2, 2]$.

Formula for the general case:

If $f(x)$ is a periodic function of period $2L$,

then $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx$,

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$,

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$.

$$a_0 = \frac{1}{4}\int_{-2}^{2} x^2\,dx = \frac{1}{4}\,\frac{1}{3}x^3\Big|_{-2}^{2} = \frac{4}{3}.$$

$$a_n = \frac{1}{2}\int_{-2}^{2} x^2 \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{1}{2}\int_{-2}^{2} x^2 \quad \frac{2}{n\pi}\, d\sin\left(\frac{n\pi x}{2}\right)$$

$$= \underbrace{\frac{1}{2}x^2\,\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{2}} - \frac{1}{n\pi}\int_{-2}^{2}\sin\left(\frac{n\pi x}{2}\right) 2x\,dx$$

$$= 2\cdot\frac{1}{2}\cdot 2^2\cdot\frac{2}{n\pi}\sin(n\pi) = 0$$

$$\int_{-2}^{2} 2\sin\left(\frac{n\pi x}{2}\right)x\,dx = \int_{-2}^{2} 2x\cdot\frac{-2}{n\pi}\,d\cos\left(\frac{n\pi x}{2}\right)$$

$$= \frac{-4x}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{2} - \int_{-2}^{2}\frac{-4}{n\pi}\cos\left(\frac{n\pi x}{2}\right) dx$$

$$= 2\cdot\frac{-8}{n\pi}\cos(n\pi) + \frac{4}{n\pi}\;\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_{-2}^{2}$$

$$= \frac{-16}{n\pi}(-1)^n$$

Similarly, we can compute b_n.

4. Solving the following pde using spectral method.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = x, \quad u_t(x,0) = 0. \end{cases}$$

Sol: Suppose $u(x,t) = X(x)T(t)$

Then, $XT'' = 4X''T$

$$\Rightarrow \frac{X}{X''} = \frac{4T}{T''} = C \text{ for some constant } C.$$

$\Rightarrow X(x) = A_1 \cos \alpha x + B_1 \sin \alpha x$ for some $\alpha, A_1, A_2, B_1, B_2$

$T(t) = A_2 \cos 2\alpha t + B_2 \sin 2\alpha t$

Since $u(0,t) = u(\pi, t) = 0$,

we may suppose $X_n(x) = \sin nx$

$T_n(t) = A_n \cos(2nt) + B_n \sin(2nt)$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(t)$$

$$= \sum_{n=1}^{\infty} \left(A_n \cos(2nt) + B_n \sin(2nt)\right) \sin nx.$$

So, $\dfrac{\partial u}{\partial t} = \displaystyle\sum_{n=1}^{\infty} \left(-2n A_n \sin(2nt) + 2n B_n \cos(2nt)\right) \sin nx.$

Since $u_t(x,0) = 0$, we have $B_n = 0$, $\forall n$.

Also, since $u(x,0) = 0$,

we have $\displaystyle\sum_{n=1}^{\infty} A_n \sin(nx) = x$.

By comparing coefficients,

we have the final result:

$$u(x,t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) \cos(2nt)$$

A function f is said to be of moderate decrease if f is continuous and $|f(x)| \le \frac{A}{1+x^2}$, for some A.

(Denote $f \in M(\mathbb{R})$)

Def: If $f \in M(\mathbb{R})$, then define the Fourier transform $\hat{f}(\xi)$ as:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi x} dx.$$

Inversion formula: $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i \xi x} d\xi.$

Properties of Fourier transform:

① $f(x+h) \overset{\mathcal{F}}{\longmapsto} \hat{f}(\xi) e^{2\pi i h \xi}$

② $f(x) e^{-2\pi i h x} \overset{\mathcal{F}}{\longmapsto} \hat{f}(\xi + h)$

③ $f(\delta x) \overset{\mathcal{F}}{\longmapsto} \frac{1}{\delta} \hat{f}(\frac{\xi}{\delta})$

④ $f'(x) \overset{\mathcal{F}}{\longmapsto} (2\pi i \xi)\hat{f}(\xi)$

⑤ $-2\pi i x f(x) \overset{\mathcal{F}}{\longmapsto} \frac{d\hat{f}(\xi)}{d\xi}$, if $xf(x) \in M(\mathbb{R})$

Let $f, g \in M(\mathbb{R})$, define $f * g(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy$

Property: $\widehat{f * g}(\xi) = \hat{f}(\xi)\,\hat{g}(\xi)$

(Rmk: $f * g \in M(\mathbb{R})$, $f * g = g * f$)

Example: Calculate the Fourier transform of $e^{-\pi x^2}$.

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i \xi x}\, dx$$

$$\frac{d\hat{f}(\xi)}{d\xi} = \int_{\mathbb{R}} (-2\pi i x)\, e^{-\pi x^2} e^{-2\pi i \xi x}\, dx.$$

$$= \int_{\mathbb{R}} i \left(e^{-\pi x^2}\right)' e^{-2\pi i \xi x}\, dx$$

$$= i\, e^{-\pi x^2} e^{-2\pi i \xi x} \Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} 2\pi \xi\, e^{-\pi x^2} e^{-2\pi i \xi x}\, dx$$

$$= -2\pi \xi \hat{f}(\xi)$$

So, we have $\frac{d\hat{f}(\xi)}{d\xi} + 2\pi\xi \hat{f}(\xi) = 0$.

$$e^{\pi \xi^2}\left(\hat{f}'(\xi) + 2\pi\xi\hat{f}(\xi)\right) = 0$$

$\Rightarrow$ $\hat{f}(\xi) = c e^{-\pi \xi^2}$ for some c.

$$\hat{f}(0) = \int_{\mathbb{R}} e^{-\pi x^2}\, dx = 1$$

$\Rightarrow$ $c = 1$

$\Rightarrow$ $\hat{f}(\xi) = e^{-\pi \xi^2}$

Example: Solve the Heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x), & x \in \mathbb{R}. \end{cases}$$

Apply Fourier Transform on both sides of the pde w.r.t x,

$$\int_{\mathbb{R}} \frac{\partial u(x,t)}{\partial t} e^{-2\pi i \xi x}\, dx = \int \frac{\partial^2 u}{\partial x^2} e^{-2\pi i \xi x}\, dx$$

$$\Rightarrow \frac{d}{dt} \hat{u}(\xi, t) = (2\pi i \xi)^2\, \hat{u}(\xi, t)$$

$$\Rightarrow \hat{u}(\xi, t) = A(\xi) e^{-4\pi^2 \xi^2 t} + C, \text{ for some } A(\xi) \text{ and } C.$$

Since $u(x,0) = f(x)$,

we have $\hat{u}(\xi, t) = \hat{f}(\xi) e^{-4\pi^2 \xi^2 t}$

Note that $e^{-\frac{x^2}{4t}} \xmapsto{\mathcal{F}} \sqrt{4\pi t}\, e^{-4\pi^2 \xi^2 t}$

So, if we define $H_t(x) = \dfrac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$.

Then, $u(x,t) = f * \underbrace{H_t}_{\text{Heat kernel}}(x)$.

Steady-state heat equation on the upper half plane:

$$\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & x \in \mathbb{R},\ y > 0 \\ u(x,0) = f(x), & x \in \mathbb{R}. \end{cases}$$

Apply Fourier transform to $\Delta u = 0$ w.r.t x,

$$(2\pi i \xi)^2 \hat{u}(\xi, y) + \frac{\partial^2 \hat{u}(\xi, y)}{\partial y^2} = 0$$

$$\Rightarrow \frac{\partial^2 \hat{u}(\xi, y)}{\partial y^2} - 4\pi^2 \xi^2 \hat{u}(\xi, y) = 0$$

$$\Rightarrow \hat{u}(\xi, y) = A(\xi) e^{-2\pi|\xi| y} + B(\xi) e^{2\pi|\xi| y}$$

Ignore the term $B(\xi) e^{2\pi|\xi| y}$,

we have $\hat{u}(\xi, y) = A(\xi) e^{-2\pi|\xi| y}$

Since $u(x,0) = f(x)$, $\hat{u}(\xi, y) = \hat{f}(\xi) e^{-2\pi|\xi| y}$

$$e^{-|x|} \xmapsto{\mathcal{F}} \frac{2}{1 + 4\pi^2 \xi^2}$$

Define $P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2}$, $x \in \mathbb{R}$, $y > 0$

Then $u(x, y) = f * \underbrace{P_y(x)}_{\text{Poisson kernel}}$.